turtles speak mathematics

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FOREWORD

This paper by Barry Newell compliments his problem-solving workbook *Turtle Confusion* (Curriculum Development Centre, Canberra, 1988). *Turtles Speak Mathematics* is directed to classroom teachers who want to understand more about using Logo in the mathematics curriculum. It is intended for secondary mathematics teachers, but should be of interest to all who are involved with mathematics.

The discussion between the Turtle and his friend EBN is refreshing and illuminating. Those who have already read *Turtle Confusion* will enjoy learning more about Dr. Newell's perspective and the ideas that he has used so successfully for teaching mathematical problem-solving. Those who have not worked with *Turtle Confusion* will find the present paper an excellent springboard for their attempts to combine mathematics and Logo.

Brent Corish
Director
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SCENE 1.

In which the Turtle reminds EBN of the importance of Logo for high school mathematics. The two friends agree to discuss the issue further.

It was mid-afternoon on a wintry Saturday. I had just finished marking a stack of trigonometry test papers when the Turtle staggered in with an arm-load of wood. He dropped the logs into the woodbox with a thump and then collapsed panting into his armchair.

'You're out of condition,' I said. 'Too much to eat and not enough exercise.'

The Turtle ignored my comment. After a while he looked over at me. 'Haven't you finished marking those tests yet?'

'Yes,' I said, 'I'm just summarising the marks. Actually, they're pretty terrible. These kids don't seem to have learnt anything at all ... I don't think I'm doing a very good job.'

'It's not easy,' said the Turtle. He hauled himself out of his chair and went over to put a log on the fire.

'I guess not,' I said, with a sigh. 'But I'm still disappointed. I work hard to make it understandable ... I try to make it interesting ... but how can you make trigonometry **really** relevant to high school kids?'

The Turtle poked the fire a couple of times and a shower of sparks went up the chimney. 'Well, at least you're thinking about it,' he said. He gave the fire another poke and then sat back on his heels, 'I've been meaning to ask you ... have you read *Mindstorms* [1] yet?'

'No, not yet ... I simply haven't had the time. But I do want to learn more about Logo ... I'll look at it during the summer holidays.'

'I think that you should read it now,' said the Turtle. He stood up and dusted off his hands.

'Why?'

'Because *Mindstorms* is not just about Logo ...it's about making mathematics relevant to learners. Papert presents Logo as an example of what can be done. You should read it.'

'But Logo is for primary school kids,' I said. 'I've heard you say that yourself lots of times.'

'I've said that primary school children can learn a lot from working with Logo ... have you ever heard me say that it **wasn't** for high school students?' said the Turtle with a smile.

'No, but something that primary kids can do is hardly going to challenge teenagers. Well, not for long ... and they've probably done it in primary school anyway.'

'You haven't really seen the range of possibilities yet,' said the Turtle. 'Logo can be used by students from primary school right through to university ... it just depends on how you use it. Sure, you have to think about Papert's ideas and figure out how they apply to your particular situation ... but his ideas are centrally important for high school teachers ... they're a lot more practical than most people believe.'

'It's one thing to talk about promising ideas ... it's another thing to put them into practice in a real classroom.'

'True ... but what about *Turtle Confusion*? We used Logo with high school students in *Turtle Confusion* ... that was successful.'

'Well ... yes, it was ... but that was different ... and the Logo puzzles were fairly easy ...'

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1 Full references are excluded from the text in order to preserve the natural flow of the conversation. Each publication referenced is identified either by its title alone (in italics) or by the name of its author alone (Seymour Papert, Margaret Donaldson, and Richard Skemp). Details of these publications are given in the bibliography.

'Nevertheless,' said the Turtle firmly, 'there **was** a lot of mathematical thinking involved ... you have to use mathematics to communicate with the turtle. You know my favourite saying ... "Turtles Speak Mathematics".'

'Uh-huh,' I said, stifling a yawn, 'I **have** heard that before ... once or twice... I've never been quite sure what you mean.' I stretched and yawned again.

The Turtle straightened up and stood with his back to the fire, warming his shell. He looked at me for a moment: 'Well, I mean that you have to use mathematical knowledge and thinking to figure out the instructions for the turtle ... and then you have to use Logo, which is a formal "mathematics-like" language, to give the turtle its instructions. I think that talking to a "mathematics speaking" robot is an unusually interesting way to use mathematics ... it opens up lots of possible activities.'

'OK,' I said. 'For the sake of argument let's agree that there are some good ideas buried in all of this ... my question is "how does a real-life, hard-working, syllabus-bound teacher make use of mathematics-speaking turtles?".'

'That's the central question,' said the Turtle. 'That's **the** central question.' He paused, lost in thought for a moment. 'Actually,' he said, grinning down at me, 'assuming that you are committed to improving your student's encounters with mathematics ... and assuming that you want to deepen your own understanding of the learning process, **and** the art of teaching, **and** mathematics itself ... and assuming that you are prepared to work hard to apply all of your new knowledge ... **then** there's nothing to it!'

'Well, I'm glad to hear that it's so easy ... we'll start first thing Monday ... out with the old, in with the new ... we teachers are always prepared to work hard at something that is guaranteed to produce good results!'

'Oh, I can't guarantee results,' said the Turtle seriously. 'Too much depends on you and ... '

'I know that you can't guarantee results,' I said, chuckling. 'But, go on. You still haven't answered my question ... what's a real teacher do in a real classroom?'

'Well, it will take a while to discuss it properly,' said the Turtle, 'and I'm ready for a cup of tea.' He walked over to the living-room door and then stopped and looked back at me: 'We could spend an hour or so on it now ... do you want to do that?'

I looked at my watch. 'Good idea. I'm game. Just the afternoon for a lively confrontation. I'll put these tests back in the study and get some pencils and paper ... why don't you make us that pot of tea?' I leant forward and began to gather my papers together.

'Gladly,' said the Turtle. 'Would you get my notes in the blue folder ... and *Mindstorms* and *Turtle Geometry* ... they're in the bookshelf beside the desk.' He headed for the kitchen.

SCENE 2.

In which a myth is dispelled and the Turtle outlines some of the connections between Logo and basic mathematical concepts. Variables make their appearance.

A little later we were again seated in front of the fire, with pencils and paper and tea and hot-buttered muffins. The Turtle was flipping through *Turtle Geometry*, stopping to read a paragraph here and there and nodding.

'Well,' he said, putting the book down on the coffee table between our chairs, 'I'd like to begin by dispelling a myth.'

'Oh, goody! A Ritual Myth-Dispelling! I can't wait!'

The Turtle looked at me over the top of his glasses: 'Really, EBN, ... '

'Dispel away,' I said, leaning forward to take a muffin.

The Turtle took a deep breath and shook his head gently. 'Where was I? Oh, yes, the "Logo is for little kids" myth ... you said before that you thought that Logo was only for primary school children.'

'Yes. I've always thought that its main application was with young children.'

'That's understandable' said the Turtle, blowing on his tea. 'Logo is used much more widely in primary schools than in high schools. It **is** excellent for young children ... they can do enjoyable things with the turtle and be developing their mathematical ability at the same time. And there is often a fundamental difference in the approach used at the two levels ... Logo is used in primary schools in a way that is close to Papert's intentions ... in high schools it's mostly taught as a computer language. In many ways, its very success in primary schools gives the impression that it belongs only in primary schools.'

'But you're saying that Logo is good for high school students?'

'Yes, and for adults for that matter. Dare I say it ... it's even good for mathematics teachers ... there's not a mathematics teacher in existence who couldn't learn something new and worthwhile from working with Logo.'

'That's easy to say,' I said, accepting the challenge. 'All you do is invent a new topic, call it "New Math" or "Turtle Geometry" or "Snake Graphics", and then you say "Hey, Everybody! Roll Up and Learn Something New!" ... it's not real mathematics.'

'What do you mean by "real mathematics"?' said the Turtle, with a twinkle in his eye. 'The mathematics of real numbers?'

'I mean things like algebra and trigonometry and coordinate geometry and calculus and estimation and statistics and ... '

'Good!' said the Turtle, rubbing his hands together and holding them out to the fire. 'You can meet all of those topics with the turtle's help ... but you need to add iteration and recursion and topology and differential equations and physics and animal behaviour and artificial intelligence and robotics and principles of design ... the list is almost endless.'

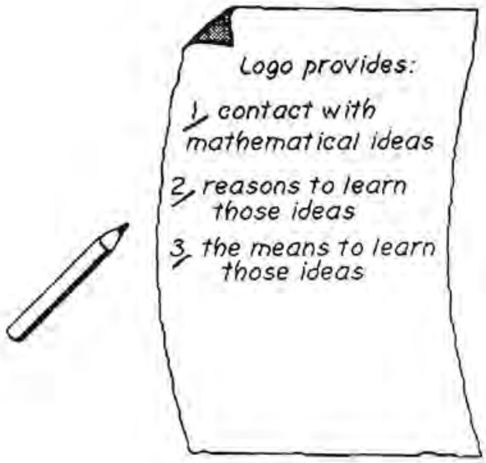
I chewed my muffin contemplatively. 'You're just listing all the things that you can do with a computer. **Any** computer language can help you to investigate those topics. All you're saying is that Logo is a computer language.'

'No,' said the Turtle. 'That's not really correct. Logo is **not** just another computer language. Of course, it **is** a computer language, and an uncommonly powerful one at that, but it's much more. It's an educational tool that's specially designed to provide an environment for **using** mathematics.'

'Yes,' I said, sipping my tea, 'you said that before. But students can have good mathematical experiences using other computer languages. I was looking at an interesting package last Tuesday ... it allowed you to type in data points and it plotted the graphs on the screen for you ... **it** was a general tool that supported exploration ... it was written in BASIC.'

'I agree that programs like that are valuable, but they simply use the computer as a super programmable calculator with graphics. Activities like that **are** good ... they're certainly better than pencil-and-paper exercises ... and you **can** use a variety of computer languages, including Logo.' He paused and then held up a finger for emphasis, 'The essential point is that Logo encourages you to **use** mathematics ... to really use it.'

The Turtle leant forward and grabbed a piece of paper and a pencil. He thought for a moment and then wrote quickly:



'There.' He handed me the paper and settled back in his chair. 'Logo supports an unusually well rounded approach. It provides connections with very important mathematical ideas, and it also provides the means to understand and learn those ideas ... and it provides reasons to want to use mathematics. A truly remarkable achievement!'

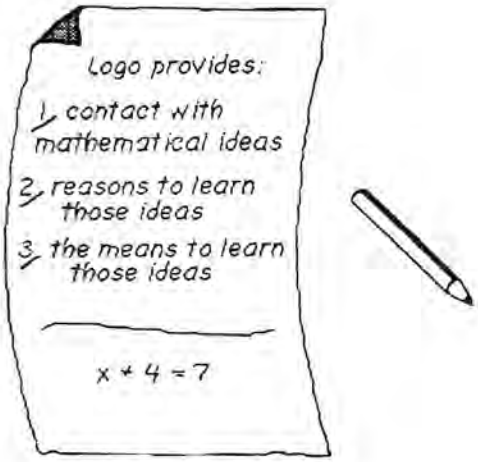
'Sorry to be so pedestrian,' I said, staring at his note, 'but I still don't see how this relates to high school mathematics. Logo involves drawing geometrical shapes and patterns ... you can't get far just drawing shapes.'

'Hopeless Case,' said the Turtle, throwing up his hands. 'Now, watch my mouth ... I'll use simple words. Logo provides a chance to use a wide range of mathematics ... not just to draw shapes. You can learn about basic concepts.'

'Basic Concepts, eh?' I sipped my tea. 'Give me an example.'

'Well ... aah ... Variables! How do **you** introduce your students to the idea of a variable?'

'Nothing to it,' I said, grinning. 'Nothing to it. I just start them off solving equations like this ... ' I dropped the paper on the table, drew a line under his words, and wrote an equation:



'Ah!' said the Turtle, the light of battle kindling in his eyes. 'So you use x in that equation as a first example of a variable?'

'Yes.'

'No!'

'Oh?'

'What can **vary** in that equation? Your x is a **constant** ... it has to be 3. That's not a good way to introduce the idea of a variable ... it's misleading ... it's ... it's wrong!'

'OK! OK! Keep your shell on! I see what you mean, but you're splitting hairs. You have to start **somewhere** ... you need to start with the simplest possible usage ... I usually start with pro-numerals and then move on to variables.'

'Yes,' said the Turtle, 'but simplicity means "simplicity", not "inaccuracy". You have to be meticulous when you simplify a concept ... it's not hard to simplify it right out of existence.'

'Yes, I can see that,' I said. 'So how do you proceed?'

'Well, it's good to start with the idea of using letters to symbolise unknown numbers ... in that context an equation like $x + 4 = 7$ is OK ... but the idea of a variable is a separate concept ... '

'Of course!' I interrupted. 'I always point out that distinction and define ... '

'Hold it!' said the Turtle, jumping up out of his chair. 'Hold it! Let's get one thing clear ... you can't **define** a new concept ... remember Skemp's discussion ... '

'Yes, but ... '

'You can't just "point out" the distinction between two different , but apparently similar, concepts and expect your students to fully appreciate what you are talking about ... learners need time to develop their own models of new concepts ... they have to work with concrete materials and then they have to ... '

'Look! I know that! I do give them lots of examples to work ... '

'What sort of examples do you use to help them grasp the dynamic nature of variables?'

'The dynamic nature of variables? Gobbledegook! What on earth do you mean by "dynamic"?'

'Well,' said the Turtle, picking up the paper and looking at my equation, 'do you agree that in this equation , $x + 4 = 7$, the x represents 3 ... it can be no other number ... it's a symbol that stands for an unspecified, but knowable, constant quantity?'

'Yes,' I said, doubtfully.

'In other words, the x in your equation stands for a number that's unchanging ... it will be the same tomorrow as it is today.'

'Can you give me a concrete example?'

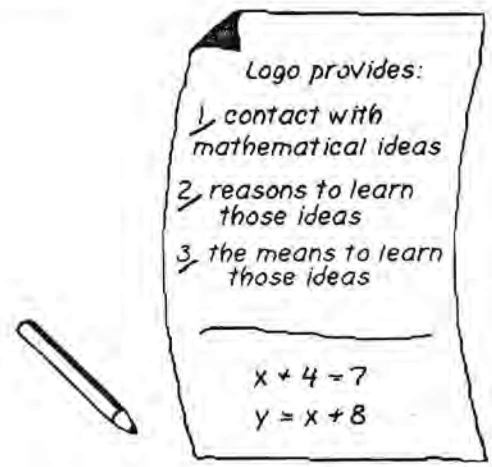
'Oh ... anything that doesn't move ... something that is stable over time. A physicist would call it "static" ... say ... the position of a rock that is lying on a table.'

'Go on.'

'Well. A variable is fundamentally different ... it represents a changing quantity ... something that varies in time ... '

'Like the position of a rock that I throw at you?' I threw an imaginary rock at him.

'Right!' said the Turtle, ducking. 'A physicist would use the word "dynamic".' He leant forward and smoothed the paper out on the table and wrote another equation:



'There, that's about as simple a variable relationship as you can get with a "pencil-and-paper" equation ... you have to use a function like that ... vary x and y varies.'

'But that's too complicated a place to start,' I said. 'My classes would have trouble if they met that equation first up.'

'I agree!' said the Turtle. 'That's exactly my point. That's why Logo is so valuable. It gives you a chance to meet variables in the right way. Logo provides **working models** of variables ... variables that you can actually use and understand ... variables that are dynamic in their effect.'

'I need another concrete example,' I said. 'You know that you can't define a new concept ... you have to exemplify it.'

The Turtle looked at me sharply and then grinned: 'Point taken.' He thought for a moment. 'Well,' he said, taking a fresh piece of paper and writing a Logo instruction, 'the inputs to the Logo commands provide a good example ... a good point of first contact.'

'Everyone understands what that instruction means,' he said[2], 'we all know what it means to tell someone to move forward a certain number of steps ... and we can all understand the difference between *FORWARD 50* and this instruction ... '

2 The Logo syntax used in this paper is that of Logo Computer Systems Incorporated (LCSI) Logo for the Apple II series of microcomputers.

I looked at the instructions for a moment. 'Sure,' I said, 'the effect of the *FORWARD* command has to change when you change the number of steps.'

'Right,' said the Turtle. 'How does that relate to variables?'

'Well ... if you say that the variable input to the Logo *FORWARD* command is a reasonable model of a mathematical variable and that ...'

'A reasonable model!' said the Turtle, sitting straight up in his chair. 'It's an **excellent** model ... it's **more** than a model ... it **is** a mathematical variable.'

'OK ... calm down ... I take your point,' I said.

The Turtle looked at me for a moment. Then he grinned, took a deep breath, and continued: 'When the input to a Logo command is changed, then the effect of the command also changes visibly ... so students can build a good working mental model of what a variable is ... just by using commands like *FORWARD*.'

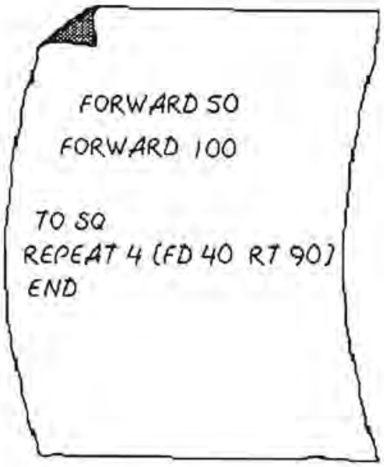
'But, saying that a different input to *FORWARD* causes a different effect on the screen is trivial ... it's so obvious that it doesn't need teaching!'

'Exactly!' said the Turtle, beaming and sinking back into the cushions. '**That's** the value of Logo ... I rest my case.'

We sat quietly for a moment. The room was getting darker and the flickering fire-light cast dancing shadows on the wall. I felt a touch of the excitement that had accompanied my early explorations of mathematics. Perhaps there was something special about Logo ... something that captured aspects of mathematics that were usually elusive.

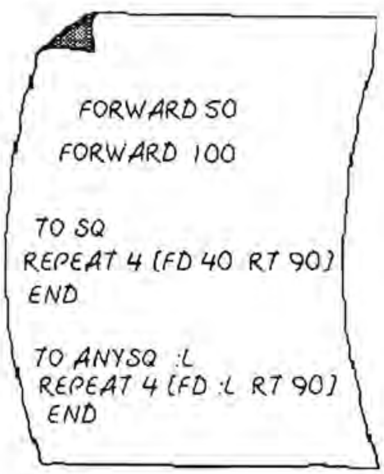
After a few minutes the Turtle roused himself and said, 'Let's take this a little further. Here ... you write down a Logo procedure that will instruct the turtle to draw a square.' He hauled himself out of his chair, pushed the piece of paper over to me, and went over to switch on the light.

I took the paper and wrote:



'Good,' said the Turtle. He came around to look over my shoulder. 'Adequate. Now write a procedure to draw a square of any size ... '

'OK,' I said, happy to show that I had learnt something from my work with *Turtle Confusion*, 'I'll call it *ANYSQ*.'



'How's that?' I said. I handed the paper to the Turtle.

'Fine,' he said. 'Now you have made the variable nature of the input quite explicit. The input to the *FORWARD* command inside your second procedure can clearly take any value ... '

'Yes!' I said, suddenly sensing where he was leading me, 'I see what you mean ... using the symbol *:L*, to stand for the variable input, introduces the idea of a symbol standing for a number that can take a range of values ... and it's dynamic ... you can actually vary the number and produce different effects on the screen!'

The Turtle beamed: 'I couldn't have said it better myself, EBN ... couldn't have said it better myself.' He trotted over to the woodbox and got another log for the fire. He paused with the log in his hands, 'Now ... can you tell me what's wrong with the two equations that we wrote down before? What's wrong with them as a pair ... from the student's point-of-view?'

I thought about that while the Turtle placed the log on the fire and then fussed about with the poker and the hearth brush.

'Well,' I began tentatively, 'I'm not sure that I see what you mean ... they're perfectly valid equations ... '

'Yes, of course they are, but taken together they are likely to be misleading ... especially to someone who is trying to understand the use of mathematical symbols and ... '

'Ah-ha!' I said, 'I've got it! The x in the first equation represents a constant and the x in the second equation represents a variable ... different concept but same symbol!'

'Yes!' said the Turtle. He went over to the table and poured himself another cup of tea. 'But it's even worse than that. The behaviour of the two equations is fundamentally different. One represents a static situation and the other represents a dynamic situation ... but that's not at all obvious from the appearance of the equations as written down. That's a major limitation of pencil-and-paper exercises. Logo provides a much better introduction ... compare your two procedures *SQ* and *ANYSQ*.'

'I see what you mean,' I said, getting more excited, 'Logo provides a really nice progression. You begin by writing a procedure that produces squares of a constant size. Then you write procedures that produce squares of any size ... and the two procedures look different and behave differently. What's more, the reason for their different behaviour is obvious ... in one case you have *40* and in the other case you have *:L*!'

'Yes,' said the Turtle, 'but the best thing about using *:L* in the square procedure is that it makes the procedure more powerful.'

'More powerful?'

'Yes. It's obvious that the procedure *ANYSQ* is more versatile than the procedure *SQ*. With *SQ* you can draw squares of only one size ... if you want to draw a square of a different size then you have to edit the procedure and change the *40* to another number ... very inconvenient. But *ANYSQ* will allow you to draw a square of any size ... all with the same procedure. *ANYSQ* is much more useful than *SQ* ... and it's more useful **because** it uses *:L* instead of *40* as the input to *FORWARD*.'

'But you still have to input a specific number when you want to use *ANYSQ* ... you still have to type *ANYSQ 40*, or *ANYSQ 70*, or whatever you want.'

'Yes.' The Turtle, put his tea cup down on the table. 'But look at it this way ... once you have defined *SQ*, or *ANYSQ*, you no longer have to think about **how** to draw a square. You have already done that once and for all ... you have isolated the "essence of squareness" ... four equal sides and 90 degree turns at the corners. The difference between the two procedures really only becomes apparent when you start using them as building blocks to construct other, more complicated, programs ... then the value of being able to whistle up a square of any size, just when you want it, becomes obvious very quickly.'

'I suppose that you could use *ANYSQ* inside another procedure that computes the size that it needs and then gets *ANYSQ* to draw each square to order?'

'Exactly!' said the Turtle. 'The step from *FORWARD 40* to *FORWARD :L* has given you the freedom to think about the next part of your problem ... about how to arrange the squares in a pattern, for example ... without having to worry about how to draw each square.

'The procedure has been **generalised** by substituting a symbol for a specific number ... and the value of doing that, the value of the generalisation, will be immediately apparent to your students.'

'Why?'

'Because it helps them to do things with the turtle,' said the Turtle patiently. 'The argument goes further. A new symbol is born every time a student creates and **names** a new procedure. Once they've had this type of experience they can really understand why algebra is important ... they have **experienced** the power of generalisation.'

We both sat quietly for a while. I handed the Turtle a muffin and took one for myself.

'That's impressive,' I said at last. 'That's a good enough example for the time being. I can see that Logo can be used to exemplify things mathematical ... and I take it that there are other standard aspects of mathematics that can be approached in the same "working model" way?'

'Yes,' said the Turtle, 'and there's more .. there are some fundamental mathematical concepts that are not usually included in high school mathematics curricula, but that should be ... and these can be modelled as well ... many of them are outlined in *Mindstorms* and *Turtle Geometry*.'

'Example?' I said.

The Turtle glanced at his watch and then looked at me speculatively for a moment. 'Well, it's getting late, but I guess we could discuss one example ... how about "state"?'

'State ... what's state?'

'Just what it sounds like,' said the Turtle. 'It means the condition that something is in right now ... anything that can change can be in different states at different times.'

'Oh, yes, I see ... like "the State of the Nation" ... sometimes it's good and sometimes it's bad.'

'Exactly,' said the Turtle. 'And how about "the state of your health"?'

'Yes ... when my health is in a poor state then I can expect my bank account to be in a poor state at the same time!'

'OK! OK!' said the Turtle, holding up his hand. 'You clearly know what "state" means. Can you apply it to the turtle?'

'Ah ... yes ... I think so,' I said, scratching my head. 'The turtle is always somewhere on the screen and it is always facing in a certain direction ... are these something to do with the state of the turtle?'

'Yes, that's right,' said the Turtle, warming to his subject. 'When you want to specify the state of the turtle you must give at least two pieces of information ... **position** information and **heading** information. Actually the turtle has other things that can change and that are part of its state description, such as whether it is visible or invisible, whether or not it draws a line as it moves, which colour it draws in, and so on ... but we can concentrate on position and heading for the moment.'

'Let me see if I've got this straight. When the turtle moves about the screen its state changes ... in other words, its position and heading changes ... I suppose that if **only** its position changes, or **only** its heading changes, then its state **still** changes?'

'Correct,' said the Turtle. He looked pleased. 'Since the turtle's state involves two quantities then both must be specified to specify the state. Position and heading, and the other parts of the state description, are called "state variables" ... they specify the turtle's state and they can vary.

'Furthermore, the turtle's state can be changed at will ... by using commands like *FORWARD* and *RIGHT*. They're called "state-change operators" ... they operate on the turtle to change its state.'

'Yes. That's clear,' I said. 'But why is the idea of state important?'

The Turtle paused for a moment: 'Because so many important things "have state" ... if we want to study anything that changes with time, anything that has properties that depend on its history, then we have to learn how to specify the state that it is in at any particular time.'

'Yes?' I said. 'Go on.'

'Well ... consider what you have to do if you want to study how and why an object changes:

'**First**, you need to be able to isolate those characteristics of the object that can change ... these are the state variables. **Second**, you need to be able to specify the present state of the object in terms of state variables... so that you can detect and describe any subsequent changes. **Third**, you need to be able to isolate those things that cause change ... in other words, the state-change operators.'

I digested that for a moment. ' So ... the ideas of state and change-of-state are of fundamental importance in the study of anything that evolves or changes in time?'

'Yes,' said the Turtle, 'any dynamic system ... you can find examples from biology, physics, chemistry, economics, and lots of other fields. The study of change, and the reasons behind change, are central issues in many subjects.'

'And that makes ways of representing change a central issue in mathematics?'

'Yes.' The Turtle, settled back in his chair. ' A very large part of modern mathematics deals with ways of representing difference and change ... variables ... functions ... calculus ... vectors ... differential equations ... '

'And all of this is discussed in *Mindstorms* and *Turtle Geometry*?'

'Correct!' said the Turtle. 'We have only scratched the surface so far.'

'Enough to draw blood though!' I said. 'You've given me good reason to believe that I can't be a high school maths teacher worth my salt and not pay attention to Logo ... I've got a feeling that I'm in for a lot of work!'

'That calls for a celebration!' said the Turtle, springing up from his chair and heading for the door.

SCENE 3.

In which the Turtle presents a parable, and introduces a captivating activity that provides a context for using mathematics.

We were in the kitchen. An aromatic stew was bubbling on the stove and the Turtle and I were sitting at the table sipping some of his famous "celebratory punch" (he kept a jug in the fridge for emergency celebrations).

'Well,' said the Turtle, getting up to stir the stew, 'I enjoyed this afternoon's discussion ... I hope that you'll do some reading now.'

'Not so fast!' I said. 'I've got my teeth into this now .. you can't escape so easily! What about your second point ... how does Logo provide **reasons** to learn mathematics?'

The Turtle turned around slowly and came back to the table. He looked at me bleakly for a moment and then brightened visibly.

'Just a minute,' he said, ' I'll get my stuff from the lounge room.' He trotted down the hall and returned in a moment with his blue folder and his books. He sat down and rummaged through his notes. 'Here,' he said handing me several sheets of paper, 'read this ... all my own work!'

'Oh?' I said, taking the pages. 'Is that good?'

'Read!' said the Turtle.

A PARABLE

Once upon a time, in a far away country, there was a community that had a wonderful machine. The machine had been built by the most inventive of their people... generation after generation of men and women toiling to construct its parts... experimenting with the individual components until each was perfected ... fitting them together until the whole mechanism ran smoothly. They had built its outer casing of burnished metal and, on one side, they had attached a complex control panel. The name of the machine, KNOWLEDGE, was engraved on a plaque set in the centre of the control panel.

The community used the machine in their efforts to understand the world and to solve all kinds of problems. But the leaders of the community were not satisfied. It was a competitive world ... they wanted more problems solved and they wanted them solved faster.

The main limitation to the use of the machine was the rate at which data could be prepared for input. Specialist machine operators, called 'predictors', carried out this exacting and time-consuming task... naturally the number of problems solved each year depended directly on the number and skill of the predictors.

The community leaders focussed on the problem of the training of predictors. The traditional method, whereby promising girls and boys were taken into long-term apprenticeship, was deemed too slow and too expensive. Surely, they reasoned, we can find a more efficient approach. So saying, they called the elders together and asked them to think about the matter.

After a few months, the elders reported that they had devised an approach that showed promise. In summary, they suggested that the machine be disassembled. Then each component could be studied at leisure and understood with ease --- the operation of the machine would become an open book to all who cared to look.

Their plan was greeted with enthusiasm. So, the burnished covers were pulled off, and the major mechanisms of the machine fell out --- they had plaques with labels like HISTORY and GEOGRAPHY and PHYSICS and MATHEMATICS. These mechanisms were pulled apart in their turn --- of course, care was taken to keep all the pieces in separate piles. Eventually, the technicians had reduced the machine to little heaps of metal plates and rods and nuts and bolts and springs and gear wheels. Each heap was put into a box, carefully labelled with the name of the mechanism whose parts it contained, and the boxes were lined up for the community to inspect.

The members of the community were delighted. Their leaders were ecstatic. They 'oohed' and 'aahed' over the quality of the components, the obvious skill that had gone into their construction, the beauty of the designs. Here, displayed for all, were the inner workings of KNOWLEDGE.

In his exuberance, one man plunged his hand into a box and scooped up a handful of tiny, jewel-like gear wheels and springs. He held them out to his daughter and, glancing at the label on the box, said:

"Look, my child! Look! Mathematics!"

I sat in silence for a while, seeing before me the face of a little girl ... the wonder in her eyes tinged with the first traces of self-doubt.

I looked up at the Turtle, who murmured, '*Reductio ad absurdum.*'

'What do you mean?'

'Well,' he said, 'we forget that gear wheels only make sense when they mesh together.

'Consider how we design our educational programs. We take the major subjects apart and reduce them to a number of main sub-sections. Then we subdivide the sub-sections. We continue until we have a large collection of little pieces that we believe that children can understand.

'As a result we present our students with disembodied fragments of subject matter ... fragments that they can't possibly make sense of ... fragments that they can't use for anything. Most of them never sense the full power of the subjects that they encounter.'

'That's clear,' I said. 'But how else can we do it? Some subjects are inherently complex.'

'Certainly they are,' said the Turtle. 'But let me ask you a question: Consider a child who has never seen the inside of an alarm clock . What would you expect to be the better learning experience for her ... taking an alarm clock apart or trying to assemble one from a pile of parts?'

'Taking one apart, of course,' I said. 'Look, I **see** what you mean ... but I don't see what we can do about it. Children can't learn major subjects intact ... surely we have to simplify them in some way?'

'I agree,' said the Turtle. 'But if we isolate each little piece of knowledge ... if we insist on artificial boundaries between subjects ... then we force our students to undergo what Margaret Donaldson calls "disembedded learning". The best approach that we can adopt is to ensure that our students encounter new ideas and methods in the context of activities that they enjoy and value. We must do all we can to emphasise the links between subjects, so that we can at least allow children to see how the gears mesh... **then** they will be able to make sense out of it all.'

'But how can you do that in mathematics?'

'In most cases the basic activity shouldn't **be** mathematics,' said the Turtle, as he got up and went across to turn off the stew, 'it should simply **involve** mathematics.

'But, the activity **does** have to be interesting and challenging. It can be a problem that the students have encountered in real-life, it can be a game or a puzzle or a task that has a strong aesthetic appeal, it can be a science or engineering project. The essential condition is that the students be put in a situation where they need to **use** mathematical ideas and reasoning to achieve goals that mean something to them.'

'And Logo can help you to do that?'

'Yes,' said the Turtle.

'I still need an example.'

The Turtle thought about it for a moment and then he came back to the table and picked up *Turtle Geometry.* ' We can use one of Ableson and diSessa's examples,' he said. 'Here ... have a look at Chapter 2.' He handed the open book over to me.

'Feedback, Growth, and Form?' I said.

'Yes,' said the Turtle. 'Look at the first part, "The Turtle as Animal".'

'I like that!' I said, chuckling. 'You always did seem like an animal to me.'

'Can we keep this serious?' said the Turtle tartly; but I suspect that he was not really offended ... he is proud of being an animal.

'OK ... sorry ... proceed.'

The Turtle drew a piece of paper toward himself and wrote a short Logo procedure. He handed me the paper:

'What's that do?' I said, frowning.

'Think it through,' said the Turtle, 'go through it step by step.'

'Well ... the procedure says to move forward 5 steps, and then ... what does *RANDOM* mean?'

'It's a standard Logo procedure ... *RANDOM n* will return a number selected at random from the range zero to *(n-1)*.'

'Oh,' I said, 'so *RANDOM 91* will return a number between 0 and 90, inclusive, each time you use it ... is that right?'

'Yes,' said the Turtle.

'OK. So the procedure says to move forward 5 steps and then turn to the right by some random amount ... the turn can be as small as 0 degrees or as large as 90 degrees ... and then *MOVE* again ... it's recursive ... the procedure calls itself and does it all again ... it just keeps going.'

'Excellent!' said the Turtle, smiling. 'A masterly analysis.'

'But what good is it?'

'Why don't you find out for yourself? Try it on the computer.'

'OK,' I said, 'I'll do it after dinner.'

' Let's eat!' said the Turtle.

SCENE 4.

In which EBN tames a Logo turtle in the pursuit of mathematical understanding. Linear equations appear in the wings.

I heard a small sound behind me and looked around quickly. The Turtle was standing in the study doorway grinning at me.

'Don't creep up on me like that!' I said sharply. 'You just took ten years off my life!'

'Sorry,' said the Turtle, still grinning, 'I just wanted to see how you were getting on ... I didn't plan to disturb you ... '

'Why aren't you in the kitchen finishing the dishes and ... '

'They're done,' said the Turtle. 'I finished about half-an-hour ago.'

I looked at my watch: 'Eight-thirty! Good grief! I've been in here for almost an hour!'

'Yes,' said the Turtle, leaning against the door jamb, 'doesn't time fly when you're having fun? ... Well, what have you been doing?'

'Mostly watching the screen ... and scratching my head.'

'What do you think of the *MOVE* procedure?'

'I hate to admit it,' I said, turning back to the computer, 'but it's fascinating. It makes the turtle move just like a small beetle ... it's genuine animation ... I'd no idea that you could do things like this with Logo.'

'OK ... sorry ... proceed.'

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'Good,' said the Turtle, 'I'm glad you feel like that ... makes my sacrifice worthwhile.'

'Sacrifice?'

'Yes,' said the Turtle. 'Doing the dishes so that you could play around on the computer ... definitely a sacrifice ... all in the interests of education.'

'Wonderful,' I said. 'What would we do without martyrs?'

The Turtle didn't answer - he knew a rhetorical question when he heard one. He left the doorway, came on into the study, and pulled a chair up next to mine. 'You were altering the procedure when I came in ... what were you trying to do?'

'Oh, I got irritated by the turtle always turning to the right ... it acted as if it was dragging a broken leg ... I wanted it to go right **and** left ... more like a real animal.'

'Good,' said the Turtle. 'Go on ... what did you do?'

'Well, I figured that I needed to use *LEFT RANDOM 91* every second *MOVE* ... but I couldn't see how to tell the turtle to use *RIGHT* one time and then *LEFT* the next time ... so I built it all into one procedure ... *RLMOVE* ... like this.'

```
TO RLMOVE
FORWARD 5
RIGHT RANDOM 91
FORWARD 5
LEFT RANDOM 91
RLMOVE
END
```

I picked up a discarded piece of computer paper, wrote down my new procedure, and handed it to the Turtle.

'Not bad,' said the Turtle, looking at the paper. 'Have you tried it out yet?'

'No,' I said. 'I was just going to when you sneaked in and scared me half out of my skin.' I leant over and typed "RLMOVE" on the keyboard and the pressed the "return" key. Immediately the turtle began to move around the screen. The Turtle and I watched the luminous trail build up for a little while and then we said, together: 'It works!'

'Fantastic!' I said, with a deep feeling of satisfaction. The turtle was scurrying around the screen turning left and right and looking for all the world as though it was exploring the area with great deliberation. Now, at least, it didn't keep tracing out right-handed curves ... its path looked much more random.

'There's a better way to do it if you want ... ' said the Turtle, pausing and looking at me with one eyebrow raised.

'Oh ... all right ... show me,' I said.

The Turtle put the paper on the table beside me and wrote quickly:

```
TO RLMOVE
FORWARD 5
RIGHT RANDOM 91
FORWARD 5
LEFT RANDOM 91
RLMOVE
END

TO MOVE2
FORWARD 5
RT 90 - RANDOM 181
MOVE2
END
```

'Neat!' I said. Then I paused. 'Ah ... how does it work?'

'Well,' said the Turtle, ' the instruction *RANDOM 181* will generate a random number from 0 to 180, inclusive ... so, when you subtract the random number from 90 you get a number between -90 and +90 ... is that clear?'

'Almost.'

'The best way to see what happens is to take some extreme examples,' said the Turtle. 'First, if the random number turns out to be 0, then the result of *(90 - RANDOM 181)* is 90 and the turtle turns right ... second, if the random number turns out to be 90, then the result of *(90 - RANDOM 181)* is 0, and the turtle doesn't turn at all ... and, third, if the random number turns out to be 180, then the result is -90 and the turtle turns left ... '

'Hang on!' I said. 'Are you saying that *RIGHT -90* is the same as *LEFT 90*?'

'Yes,' said the Turtle, 'Logo commands like *FORWARD*, *BACK*, *RIGHT*, and *LEFT* will all take real inputs ... including negative numbers and decimal numbers ... '

'So, *BACK -100* makes the turtle go **forward** 100 steps?'

'Yes ... and you can include calculations if you want to ... *FORWARD 20.382 + 29.618*, *FORWARD 2*25*, and *FORWARD 125/2.5* will all give you the same result as *FORWARD 50* ... and so do combinations like *FORWARD 20 FORWARD 30* and *FORWARD 30 FORWARD 20* ... the instructions themselves are commutative.'

'Ah-ha!' I said, with sudden insight. 'So Logo provides a working model of arithmetic ... you can actually play around with it ... you can poke it and see how it jumps!'

'Exacto!' said the Turtle, jumping up and knocking his chair over. He tottered for a second and then fell flat on his back. He lay there kicking his legs in the air and rocking from side to side.

I kept a straight face as I reached over and pulled him to his feet. He picked up the chair and, looking embarrassed, seated himself beside me again.

'Where were we?' he said quietly. 'Oh, yes ... Logo provides a working model of arithmetic.'

'Yes, I can see that ,' I said. I began chuckling to myself.

The Turtle went on in a dignified manner: 'But there's an even more fundamental point than that ... Logo allows you to experiment . You can try out your ideas and see if you get the result that you expect ... in fact, you can't avoid doing it that way ... there is always the chance that you will have bugs in your thinking and the turtle will ferret them out.'

'Turtles ferreting out bugs? What kind of animalistic mixed metaphor is that?' I said, still trying not to laugh.

'You know what I mean,' said the Turtle stiffly. 'If you make a mistake in expressing your ideas in their mathematical form ... that is, as Logo instructions ... then the turtle will do something that you don't expect. So, you'll quickly find out that you have made an error, or that your original idea was not adequate. You don't need anyone else to tell you. That's an essential characteristic of Logo. The "result" of a Logo computation is a visible movement ... you can **see** the result and you can assess it yourself.'

'Yes,' I said, 'that's clear ... but what did you mean by "expressing ideas in mathematical form"? ... what's that got to do with Logo instructions?'

'Good question,' said the Turtle, relaxing a little. 'If you want to use mathematics to solve a problem, then the first step is to express the problem in mathematical terms ... right?'

'Yes.'

'Well, it's the same with Logo. If you want the turtle to behave a certain way on the screen, then you have to go through several distinct steps:

'First, you have to decide how the behaviour can be described, that's your "theory";

'Second, you have to write down a description of your theory in the formal Logo language, so that the turtle will understand your ideas;

'Third, you run the Logo program to see if your "theory" is right.

'Fourth, you need to correct any "bugs" in your theory ... and run the whole process again.

'The second step ... writing your "theory" down in formal terms ... involves **exactly** the same process as that needed to express a problem in mathematical terms. Writing Logo programs, and modifying them to correct errors or to improve your theories, is exactly the same as **doing** mathematics.'

'Is that why you always say that you can't "do" Logo without "doing" mathematics at the same time?' I said.

'Yes.'

'So **that's** why you didn't need to build a lot of obvious mathematics into *Turtle Confusion*!'

'Exactly!' said the Turtle. 'I didn't need to because the use of the Logo language guaranteed that everyone would have to "do mathematical things" if they worked with the *Turtle Confusion* puzzles.

'But there was more to it than that ... I wanted people to have the experience of discovering the mathematics that was involved ... I couldn't discuss the mathematics without taking away their opportunity to make the discoveries themselves. That's why the hints and suggestions are framed as riddles with built-in verification ... so that the whole process of theory-formation, and prediction, and observation could be made more explicit ... '

'More explicit ... but still with no answers given!'

'Yes,' said the Turtle, chuckling.

Then he became serious again: 'Actually, the absence of answers is a central point. Our students must develop the ability to test their own ideas. They need opportunities to build their own theories, to express those theories in appropriate language, and then to test them out for themselves ... they must become critical of their own thinking if they are to become successful problem solvers in real life.'

'That's exactly what I was doing with the *MOVE* procedure!' I said, with rising excitement. 'I had a theory about how to make the turtle behave in a more realistic way ... and I had to express it in Logo "mathematics" ... and I tested it out for myself!'

'Yes,' said the Turtle, 'that's a good example ... the best examples come from your own experiences.' He paused. ' Do you want to go on? Do you want to take another step with the *MOVE* procedure?'

'Yes, I do,' I glanced back at the turtle wandering around the screen, 'I really enjoy this.'

'Well,' said the Turtle after a moment's thought, 'you've already altered the *MOVE* procedure to make the turtle turn both left and right. That was a worthwhile improvement. Now I want you to give the turtle some intelligence.'

'Intelligent turtles!' I said, 'What will they think of next!'

'Shall we go on?' said the Turtle, looking a little tight around the mouth.

'Yes.'

'Good ... now ... so far the turtle has been wandering about at random. I want you to set up a barrier so that it stays in the left-hand half of the screen all the time ... it's not allowed to come over into the right-hand half.'

I thought about barriers for a while and then asked, 'If I draw a vertical line right across the screen ... to separate the left and right sides ... can I get the turtle to detect when it has touched the line?'

'Good question,' said the Turtle, relaxing again. 'But, I'm afraid not ... in some versions of Logo the turtle can be programmed to detect when it has touched a line, or to detect changes in the underlying background colour of the screen, but we can't do it with the Logo that we have here. You need to think of some other approach.'

'Well, I'm going to draw a line anyhow ... to mark the barrier.' I stopped the *MOVE* procedure, cleared the screen, and instructed the turtle to draw a vertical line through the *HOME* position.

'Good,' said the Turtle. 'Now ... how are you going to get the turtle to detect the barrier?'

'Well,' I said, 'is it possible to find out where the turtle is on the screen ... where it is exactly ... its position?'

'Great! Excellent question! You really are sharp tonight!' said the Turtle, bouncing up and down on his chair.

'Watch it! You'll end up on your back again!'

The Turtle must have regained his sense of humour because he grinned at me ... but he stopped bouncing. 'Yes,' he said, 'you **can** ask the turtle to tell you where it is on the screen. You use the reporters *XCOR* and *YCOR* to get the turtle's x-coordinate and y-coordinate values separately, or the reporter *POS* to get the turtles x- and y-coordinates together.'[3]

'So there must be a coordinate grid overlaid on the screen ... how is it defined?'

'The centre of the screen has coordinates $x = 0$ and $y = 0$, as you might expect. And x increases to the right, in the normal way, and y increases up the screen ... just like conventional Cartesian axes ... the units are "turtle steps".'

'Well,' I said, 'then the turtle's x-coordinate will be negative whenever it is in the left-hand half of the screen ... so all you need to do is find out when the turtle's x-coordinate becomes zero or positive ... and stop it moving when that happens.'

'Close,' said the Turtle. 'But we want the turtle to keep moving ... we don't want it to stop as soon as it touches the barrier.'

'OK, then we turn it back when it tries to enter the forbidden region ... we could tell it to turn right by 180 degrees whenever it hits the barrier.'

'That might work,' said the Turtle slowly.

'Would you mind telling me how to modify the *MOVE2* procedure,' I said.

'I suppose not,' said the Turtle, looking at his watch. 'You really should think about it yourself, but we can't stay here for ever ... nearly time for a snack ... '

He picked up another sheet of computer paper and wrote out a new version of the *MOVE2* procedure. He called it *MOVELEFT*:

```
TO MOVELEFT
RIGHT 90 - RANDOM 181
FORWARD 5
IF XCOR > 0 [RT 180 FD 10]
MOVELEFT
END
```

3 A 'reporter' is a Logo primitive that returns information, rather than commanding the turtle to carry out an action. A reporter essentially asks the turtle a question. See, for example, the manuals written by Logo Computer Systems Incorporated to support LogoWriter.

The Turtle handed me the paper and sat in silence as I read it through.

'How does the *IF* statement work?' I said.

'Actually, it's an *IF-THEN* statement ... **if** the condition is true ... that is, if *XCOR* **is** greater than zero ... **then** all of the commands inside the square brackets are carried out ... if the condition is **not** true, then the commands inside the brackets are ignored.'

'Fair enough,' I said, putting the paper down beside the keyboard. 'What's the *FORWARD 10* for?'

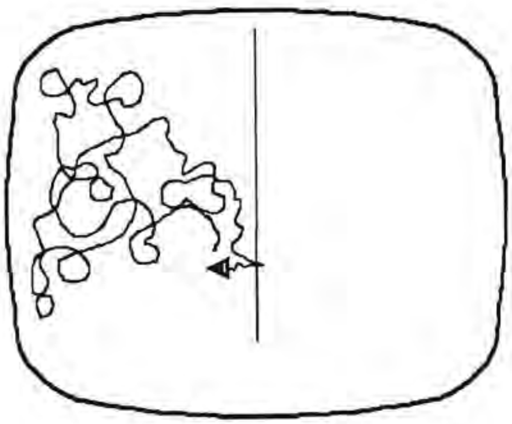
'Oh, I just thought that we probably should give the turtle a kick back into the left-hand side of the screen whenever it crosses the barrier ... let's try it out and see how it goes.'

The Turtle sat back as I typed in the new procedure and started it running. At first the turtle wandered off around the left-hand side of the screen ... it did not venture near the centre-line for quite a while ...

'Seems like it knows,' I said softly.

'Yep,' said the Turtle, 'it sure does ... turtles have a sixth sense.'

The turtle drifted towards the centre of the screen and then suddenly it spun around and leapt back to the left.



'Fantastic!' I said. 'Looks like the turtle got stung!'

'Mmm,' said the Turtle, 'maybe the *FORWARD 10* is a bit vicious ... but you can play with that later.'

The left-hand side of the screen was filling up with contorted turtle-trails and it was now clear that the procedure was working quite well. The Turtle stood up and carefully moved his chair back to its normal place. 'Shall we call it quits?' he said. 'You can continue later if you want to, but I'm getting cold ... let's go back to the fire.'

'I could sit here all night,' I said. 'It's like watching gold-fish ... '

The Turtle licked his lips: 'Actually ... talking of fish ... it **is** time for a snack.'

I can take a hint, especially when food is involved. I turned off the computer and followed him along to the kitchen.

SCENE 5.

In which the Turtle and EBN discuss the importance of intuitive knowledge as a prerequisite for mathematical learning. EBN resolves to introduce Logo to his mathematics classes.

'Well,' said the Turtle, when we had settled down once more in front of the fire, 'what was your impression of our "turtle as animal" exercise?'

'Really captivating,' I said, reaching for a slice of toast topped with grilled sardines and cheese. 'I can see all sorts of possibilities for projects based on the *MOVE* procedure ... and I see what you mean by "giving the turtle some intelligence" ... the way you program it to react when it hits a barrier, or encounters a problem of some sort, needs to be as intelligent as possible.'

'That's right,' said the Turtle. 'But the important question is ... would your students enjoy setting-up the turtle in that way?'

'Definitely! My own attitude toward the turtle changed because of the experiments that we just did ... it seems more like a real animal now ... I even feel quite fond of it . Yes, I think that most of my class would become very involved in the problem of programming the turtle to behave intelligently.'

The Turtle looked pleased. 'Of course, it can be taken a lot further,' he said, 'you can set up barriers of different types ... make the turtle stay inside a square ... or a circle ... make sure that its nose does not cross the barrier ... make it run a maze. These activities lead straight into robotics ... there are lots of good projects.'

'And all of it involves "real" mathematics?'

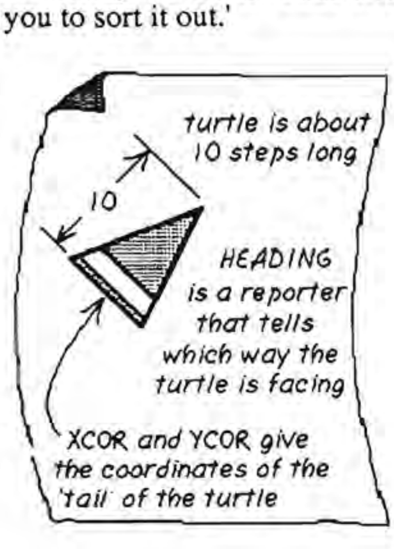
'It certainly does!' said the Turtle. 'The introduction of precise measurements of position, *via XCOR* and *YCOR*, opens the door to lots of your "real" mathematics ... linear equations, coordinate geometry, trigonometry ... '

'Trigonometry! Give me an example!'

'I'll just suggest a project,' said the Turtle. 'Did you notice that the turtle crossed over into the right-hand part of the screen when we were running the *MOVELEFT* procedure?'

'Yes ... I was thinking about that before ... it was very messy having the turtle cross the line before it was turned around and kicked back to the left ... it didn't look right.'

'Well,' said the Turtle, 'you can tidy that up. There's an easy way to do it roughly. But you need to use trigonometry to do it really neatly ... you can make the tip of the turtle's nose the sensitive point. I'll leave you to think about it.' He reached over for a piece of paper, sketched a screen turtle, and then added some dimensions and notes. 'There,' he said. 'That should help you to sort it out.'



'Thanks,' I said, with a grimace. I looked at his sketch for a moment and then dropped it on the table. 'I'll have a go at that tomorrow.'

'Actually,' I continued, 'I want to ask another question before I forget. The turtle's movements seemed really random to me ... how good is the *RANDOM* procedure? ... can you use it to study random processes? ... can you study things like the "drunkard's walk" using the *MOVE* procedure?'

'Yes,' said the Turtle, 'you can use the wandering turtle to study the properties of the "random walk" ... you can develop an intuitive feel for quite a number of the basic concepts of probability and statistics that way ... but the turtle can also help you derive quantitative results. For example, you can use *XCOR* and *YCOR* to keep track of the turtle's movements over the coordinate grid and to calculate how far it has moved from its starting point.'

He reached for a helping of grilled sardines as I got up to pour each of us another cup of tea. Eventually the Turtle said, 'The main point that I wanted to make ... with the *MOVE* procedure ... was that there are activities that can be carried out with Logo that are highly motivating ... such activities may not be primarily mathematical, but they do involve mathematical ideas and thinking and they provide good **reasons** for learning the mathematical ideas embedded in Logo.'

'It's all a matter of "turtle control" ... isn't it?' I said.

'Yes! You've hit the nail on the head!' said the Turtle. 'That's a good way to express my second point! That's **exactly** why Logo makes it worthwhile to learn mathematical ideas ... you need to understand those ideas so that you can exercise better, more versatile, control over the turtle.'

'One of my fondest dreams,' I said, putting my tea-cup on the table.

'What is?'

'Better turtle control.'

The Turtle showed no outward signs of emotion ... he merely reached for some more grilled sardines.

I picked up *Turtle Geometry* and read a little more of Chapter 2 ... modelling smell ... modelling sight ... predator and prey ... there were lots of exciting ideas. Clearly, if you could control the turtle you could do interesting things, and to control the turtle you needed to understand the mathematics of turtle control. Logo **did** provide better-than-usual reasons to learn mathematical ideas. I was convinced. I looked over at the Turtle.

'Two down,' I said, holding up two fingers. 'One to go.'

'What do you mean?'

'This afternoon you said that Logo provided "**contact** with mathematical ideas", and you demonstrated what you meant by that and convinced me ... and you said that Logo provided "**reasons** to learn those mathematical ideas", and you have just demonstrated what you meant and convinced me. That leaves your third point ... Logo provides "the **means** to learn those mathematical ideas" ... care to demonstrate and convince?'

The Turtle thought for a while and then he said, 'I'm going to have to talk about learning ... is that OK?'

'I guess so,' I said, looking at my watch, 'as long as we aren't here past midnight.'

The Turtle gave me a hard look, and I looked innocently back ... we had an ongoing argument about how people learn, and it didn't take much for our discussions of this topic to get rather heated.

'We did agree that everyone learns by building mental models of the things that they experience ... didn't we?' the Turtle began, tentatively.

'Yes,' I said, 'we did ... at last count.'

'Well, it seems obvious to me that, roughly, these models can be divided into two types ... those that the learner is aware of, because they have been thought about to some extent, and those that the learner is unaware of ... '

'Hogwa ... er ... how can you possibly construct a mental model and be unaware of it?' I said, trying not to slip automatically into my usual sceptical approach.

'We do it all the time,' said the Turtle. 'At least, those of us with functioning minds do it all the time.'

'Come on ... let's keep this civilised,' I said. 'How can you build a model of an experience and not know that you have done it? ... give me an example.'

The Turtle scratched the back of his head and pushed his glasses up on his nose: 'Well,' he said, 'consider the models that you have of the relationship between the expressions on your friends' faces and the way that they feel at the moment ... these are very complex models, that you use every time you meet another person, but you are not usually aware of them.'

'I guess you're right,' I said, reluctantly. 'Even young children must have models like that ... even before they can talk they are good at judging their parent's moods ... they must have excellent models.'

'That's a good example of an example,' said the Turtle with some enthusiasm. 'But you should be able to think of other examples fairly easily ... shall I go on?'

'Yes, I suppose so' I said. I looked at the last piece of toast but decided against it.

The Turtle paused to collect his thoughts: 'Well ... we can divide our mental models up in another way ... into those we think about formally and those that we don't think about formally.'

'By "formally" you mean ... '

'By "formally" I mean models that we can express clearly and systematically *via* natural language, or other systems of symbols, and that we can manipulate deliberately following a set of rules.'

'Manipulate deliberately?'

'Yes,' said the Turtle, 'you remember what I was saying last week ... about the need for our mental models to be **working models** ... you can't make predictions on the basis of static models ... you have to be able to run them in your mind if you want to predict events.'

'OK,' I said, 'I accept that ... go on.'

'A mathematical model of a physical system is a good example of a formal model ... once you have described the system in mathematical terms, you can manipulate the model deliberately, using the formal rules of mathematics, to deduce the way that the real system will respond to changes in its environment.'

'I see that,' I said. 'But there must be mental models that you can't manipulate formally but that still allow you to make good predictions.'

'Exactly!' said the Turtle. 'That's precisely what I'm getting at. The best examples are those models that we are unaware of ... they are often models that describe complex activities that would be very hard to express in formal terms.'

'Like our models of the relationship between facial expression and emotion?'

'Yes,' said the Turtle, 'you have to agree that you certainly don't think about those models formally.'

'Yes, I see what you mean ... I can't write down formal reasons why I interact with people the way I do ... but I can usually predict the consequences of my actions very well.'

'Right!' said the Turtle. 'What's the word for that type of informal, but highly effective knowledge? ... give me one word!'

I looked at him blankly.

'Intuition!' he said, holding out his hands. 'INTUITION.'

'Oh,' I said weakly. 'Intuition.'

'Yes,' said the Turtle, 'we often have working mental models that we **can't** express formally, either because we're unaware of them, or because the models are too complex ... but we can **still** manipulate those models to provide reliable predictions ... when we do that we say that we are "reasoning intuitively" ... '

He stopped and looked at me. I must have looked unconvinced, because he said, 'don't you agree?'

'I don't disagree ... I'm just trying to keep up with you.'

'Do you want me to explain it again?'

'No! Please don't!' I said. 'We can go over it again tomorrow if necessary ... I've got the general drift of your argument. Tell me how this relates to Logo and mathematics.'

'*Via* the turtle.'

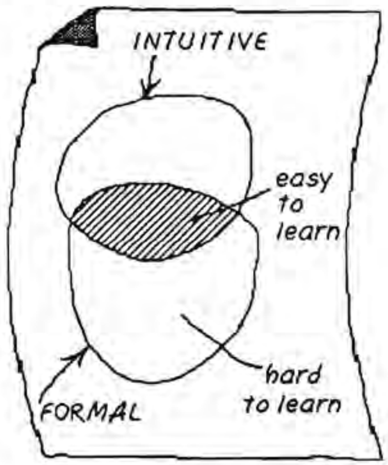
'Surprise!'

The Turtle looked at me sideways, but said nothing. Then he leant forward in his chair, took pencil and paper, and drew a diagram:

'This line represents the boundaries of an area of knowledge that you understand intuitively,' he said, 'and this one represents the boundaries of an area of formal knowledge.' He drew another line overlapping the first.

'The basic idea behind the use of the turtle in Logo is that you can learn formal ideas relatively easily if they overlap with intuitive ideas that you have already.' He shaded in the overlap region and labelled it.

'Formal ideas that are entirely new ... that can't be built on an intuitive foundation ... are very much harder to grasp.' He added another label and then passed the diagram across to me:



'OK,' I said, 'that's clear enough. Continue ... where does the turtle come in?'

'Well,' said the Turtle, 'Papert calls the turtle a "transitional object" because it helps people to establish links between things that they know well intuitively and mathematical ideas that they don't yet know well.'

'Go on.'

The Turtle leant forward and took the last piece of toast. 'Look at it this way,' he said, licking the butter off his fingers. 'Turtles are easy to identify with because they are animals.'

'How can that have anything to do with mathematics?' I said. I got up and went over to stir the embers in the fireplace.

'Well ... an animal, or a person, or a car, has a definite location in space and it is always facing in a certain direction ... in other words, it has a position and a heading. A screen turtle, or a floor turtle, also has position and heading.'

'You mean that position and heading are state variables for all of those objects,' I said casually, straightening up and leaning against the mantlepiece.

'Yes.' He stopped and looked up at me. 'Hey! Not bad for a beginner!'

'You will note that your words are not wasted on me,' I said with a smile. 'But, how does all of this help us?'

'Well, the turtle is a real object ... a concrete object that exists in the real world ... you can see it and you can control it . And the turtle speaks mathematics. That is, the instructions that you give the turtle must be expressed in a language that it "understands" ... and this language ... Logo ... has been designed so that the approach needed to control the turtle involves ideas that satisfy two criteria .. one, you already understand them intuitively ... and, two, they are fundamental mathematical ideas.'

'Hang on,' I said, 'let's see if I've got this right ... you are saying that I've got a strong intuitive understanding of the meaning of the words "go forward 50 steps" and, **because of that intuition**, I can easily understand the *FORWARD* command and the effect of changing the input value. But, at the same time, the *FORWARD* command provides a working example of a powerful mathematical idea ... so I can bring my intuition to bear on the problem of how to understand that mathematical idea.'

'Yes,' said the Turtle. 'You and the turtle have state variables in common ... that means that you can use all of the intuitive knowledge that you have concerning the movement of your own body in space to understand the problem of moving the turtle around on the screen ... **and**, because Logo provides a formal language for the description of movement in space, you can reverse the process ... you can use what you learn about turtle control to help you understand, **formally**, the way you yourself move around. In other words, the mathematical ideas that underlie the turtle's movements are the same ideas that underlie all movement of bodies in space ... so there are strong links to physics.'

'I see what you mean!' I said. 'The process is like leap-frog ... you can build your formal knowledge of Logo and mathematics on the basis of your intuitive knowledge of body movement .... and then that new formal knowledge, once you have played around with it, becomes intuitive knowledge itself ... and forms a foundation for learning about additional areas of mathematics and physics.'

The Turtle looked at me in amazement. 'That's right,' he said, 'That's exactly right ... you are really starting to put it together.'

I laughed: 'Thought I wasn't going to make it, eh?'

'Well,' said the Turtle, quietly, 'I must admit that I have entertained a few doubts from time to time ... but mostly about my ability to describe my own thoughts clearly.'

'You do fairly well,' I said, smiling at him, 'especially since I sometimes go out of my way to be difficult.'

'Yes, I had noticed that. Seriously, though, I really believe that most high school mathematics teachers already have the knowledge needed to use Logo effectively. But, it takes an ability to see the old mathematics from a new perspective ... and it's not necessarily easy to develop that perspective.'

'You seem to expect teachers to learn a lot.'

'Well, I believe that they can do it. Besides, their students are expected to learn a lot.'

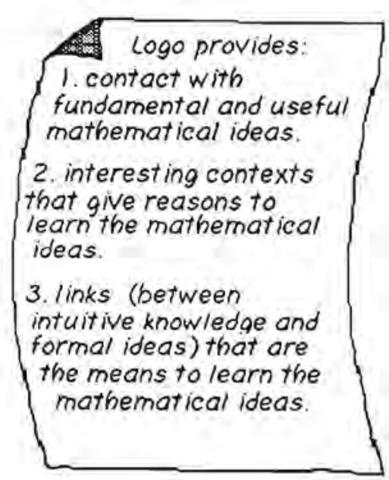
'Point taken,' I said with a grin. 'I guess it's OK ... provided we get some help.'

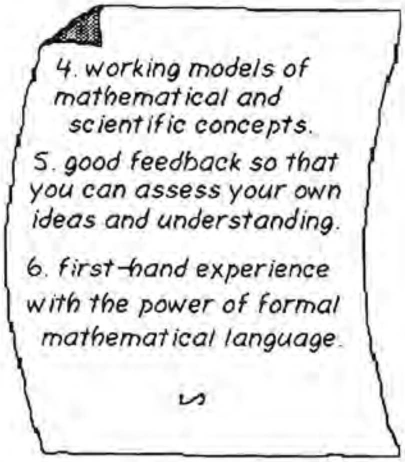
'Try to escape!' said the Turtle. He looked at his watch. 'Enough?'

'Yes ... enough,' I said. 'But let's summarise the major points before we stop.'

I went over to the table for some pieces of paper and took them back to the mantelpiece. I wrote down the Turtle's original points, with some amplification: 'I now understand what you mean by these three points. But I want to add some of my own.' I took a second sheet of paper and added two extra statements.

I passed the pages over to the Turtle. He read them through a couple of times. 'Yes,' he said, 'not bad ... I have one more to add ... '. He wrote an extra line and handed the pages back to me:





'Great!' I said. 'That's a good summary ... let's call it quits for the moment. I'll have to think this through before I can absorb any more. Thanks for being so insistent ... and so patient.'

'You're welcome,' said the Turtle, looking pleased.

For a while the room was quiet. We had covered a lot of ground in a short time and some of the ideas were still not clear in my mind. The actual implications for teaching were going to take a while to assess ... but I was sure that I needed to re-examine my approach to mathematics. It was clear to me now that it is essential for students to meet mathematics in contexts that are meaningful to them ... contexts where they already have developed strong intuitions ... contexts where the use of mathematics gives them power. And it seemed that Logo provided a very good example of the way that this could be arranged in the classroom.

The Turtle stood up and stretched. 'Well,' he said, 'what do you think?'

'I begin to see that there **is** another way to approach high school maths teaching. I'm starting to see all kinds of possibilities ... and all kinds of problems.'

'Good,' said the Turtle, as he began to collect the cups and plates together, 'at least we are on the same wavelength ... at least we are talking the same language now.'

'Yes,' I said. 'Mathematics!'

FURTHER STEPS

Logo represents an important, but largely overlooked, opportunity for high school mathematics teachers. I hope that *Turtles Speak Mathematics* will stimulate you to investigate this "Logo opportunity" further. Try out the *MOVE* procedure for yourself. If you have had no previous contact with Logo then find a friend who can help you get started.

The following books will support you in your exploration of the ideas embodied in this paper:

Abelson, H., and diSessa, A. A., 1980, *Turtle Geometry: The Computer as a Medium for Exploring Mathematics*, (The MIT Press).

Donaldson, M., 1978, *Children's Minds*, (Fontana).

Newell, B., 1988, *Turtle Confusion*, (Curriculum Development Centre).

Papert, S., 1980, *Mindstorms: Children, Computers, and Powerful Ideas*, (The Harvester Press).

Skemp, R. R., 1971, *The Psychology of Learning Mathematics*, (Penguin).

If you want to learn about the Logo language itself then I recommend:

Abelson, H. 1982, *Logo for the Apple II*, (BYTE/McGraw-Hill).

Martin, D., Prata, S., and Paulsen, M., 1984, *Apple Logo Programming Primer*, (SAMS).

Thornburg, D. D., 1983, *Discovering Apple Logo: An Invitation to the Art and Pattern of Nature*, (Addison-Wesley).

Watt, M., and Watt, D., 1986, *Teaching With Logo: Building Blocks for Learning*, (Addison-Wesley).

Finally, if you are already familiar with Logo but want to extend your knowledge, in addition to reading *Turtle Geometry* by Abelson and diSessa, you will find the following two books useful:

Harvey, B. 1985, *Computer Science Logo Style*, (The MIT Press).

Thornburg, D. D., 1986, *Beyond Turtle Graphics: Further Explorations of Logo*, (Addison-Wesley).

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